

Find the general solution of $y''' - y'' + 4y' - 4y = 20\cos 2t$. (2) EACH EXCEPT AS NOTED SCORE: ____ / 40 PTS

$$\begin{aligned} r^3 - r^2 + 4r - 4 &= 0 & r &= \pm 2i, 1 \\ r^2(r-1) + 4(r-1) &= 0 & \rightarrow & y_h = c_1 \cos 2t + c_2 \sin 2t + c_3 e^t \\ (r^2 + 4)(r-1) &= 0 \end{aligned}$$

$$y_p = At \cos 2t + Bt \sin 2t \quad (4)$$

$$y_p' = A \cos 2t - 2At \sin 2t = (2Bt + A) \cos 2t + (-2At + B) \sin 2t \quad (4)$$

$$+ 2Bt \cos 2t + B \sin 2t$$

$$y_p'' = 2B \cos 2t + (-4Bt - 2A) \sin 2t = (-4At + 4B) \cos 2t + (-4Bt - 4A) \sin 2t \quad (4)$$

$$+ (4At + 2B) \cos 2t - 2A \sin 2t$$

$$y_p''' = -4A \cos 2t + (8At - 8B) \sin 2t$$

$$+ (-8Bt - 8A) \cos 2t - 4B \sin 2t \quad (4)$$

$$= (-8Bt - 12A) \cos 2t + (8At - 12B) \sin 2t$$

$$-y_p'' + (4At - 4B) \cos 2t + (4Bt + 4A) \sin 2t \quad (4)$$

$$+ 4y_p' + (8Bt + 4A) \cos 2t + (-8At + 4B) \sin 2t$$

$$- 4y_p - 4At \cos 2t - 4Bt \sin 2t$$

$$= (-8A - 4B) \cos 2t + (4A - 8B) \sin 2t = 20 \cos 2t \quad (4)$$

$$\begin{array}{l} -8A - 4B = 20 \\ 4A - 8B = 0 \end{array} \rightarrow \begin{array}{l} 2A + B = -5 \\ A - 2B = 0 \end{array} \rightarrow \begin{array}{l} A = -2 \\ B = -1 \end{array}$$

$$y = -2t \cos 2t - t \sin 2t + c_1 \cos 2t + c_2 \sin 2t + c_3 e^t$$

Find the general solution of $x^2 y'' - xy' + 5y = x \csc(2 \ln x)$. $g = \frac{x \csc(2 \ln x)}{x^2} = \frac{1}{x \sin(2 \ln x)}$ SCORE: ___ / 30 PTS

$$r^2 - 2r + 5 = 0$$

$$r = 1 \pm 2i$$

$$y_h = C_1 x \cos(2 \ln x) + C_2 x \sin(2 \ln x)$$

② EACH EXCEPT AS NOTED

$$W = \begin{vmatrix} x \cos(2 \ln x) & x \sin(2 \ln x) \\ \cos(2 \ln x) - 2 \sin(2 \ln x) & \sin(2 \ln x) + 2 \cos(2 \ln x) \end{vmatrix} = 2x \text{ ④}$$

$$y_p = -x \cos(2 \ln x) \int \frac{1}{x \sin(2 \ln x)} \frac{x \sin(2 \ln x)}{2x} dx \text{ ④}$$

$$+ x \sin(2 \ln x) \int \frac{1}{x \sin(2 \ln x)} \frac{x \cos(2 \ln x)}{2x} dx \text{ ④} \leftarrow u = \sin(2 \ln x)$$

$$du = \frac{2 \cos(2 \ln x)}{x} dx$$

$$= -x \cos(2 \ln x) \cdot \frac{1}{2} \ln|x| + x \sin(2 \ln x) \cdot \frac{1}{4} \ln|\sin(2 \ln x)| \text{ ④}$$

$$= -\frac{1}{2} x \ln|x| \cos(2 \ln x) + \frac{1}{4} \ln|\sin(2 \ln x)| \sin(2 \ln x)$$

$$y = \underline{-\frac{1}{2} x \ln|x| \cos(2 \ln x) + \frac{1}{4} \ln|\sin(2 \ln x)| \sin(2 \ln x)}$$

$$+ C_1 x \cos(2 \ln x) + C_2 x \sin(2 \ln x)$$

$y = e^{-x}$ is a solution of

$$xy'' + (2x+1)y' + (x+1)y = 0.$$

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$y = 3x^2 - 4x$ is a particular solution of

$$xy'' + (2x+1)y' + (x+1)y = 3x^3 + 11x^2 - 4.$$

Solve the initial value problem

$$xy'' + (2x+1)y' + (x+1)y = 8 - 22x^2 - 6x^3, \quad y(-1) = 0, \quad y'(-1) = 0.$$

$$y_2 = ve^{-x}$$

$$y_2' = v'e^{-x} - ve^{-x}$$

$$y_2'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}$$

$$xy_2'' + (2x+1)y_2' + (x+1)y_2$$

$$= e^{-x} [xv'' - 2xv' + xv + (2x+1)v' - (2x+1)v + (x+1)v]$$

$$= e^{-x} (xv'' + v') = 0$$

$$\text{LET } u = v' \quad x \frac{du}{dx} + u = 0$$

$$\frac{1}{u} du = -\frac{1}{x} dx$$

$$\ln|u| = -\ln|x| \quad \textcircled{1}$$

$$u = x^{-1}$$

$$v' = x^{-1}$$

$$v = \ln|x|$$

$$y_2 = e^{-x} \ln|x|$$

$$y = 8x - 6x^2 + c_1 e^{-x} + c_2 e^{-x} \ln|x|$$

$$y' = 8 - 12x - c_1 e^{-x} - c_2 e^{-x} \ln|x| + c_2 x^{-1} e^{-x}$$

$$y(-1) = -14 + c_1 e = 0$$

$$y'(-1) = 20 - c_1 e - c_2 e = 0$$

$$c_1 = \frac{14}{e}$$

$$20 - 14 - c_2 e = 0 \rightarrow c_2 = \frac{6}{e}$$

$$y = 8x - 6x^2 + 14e^{-x-1} + 6e^{-x-1} \ln|x|$$

② EACH
EXCEPT AS NOTED

Use elimination (as shown in lecture) to solve the system

② EACH EXCEPT AS NOTED

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$$(3D+2)[x] + 2D[y] = 2t+1$$

$$(D+6)[x] + (D+2)[y] = 2t+1$$

$$\Rightarrow \begin{cases} (D+2)(3D+2)[x] + (D+2)(2D)[y] = (D+2)[2t+1] = 2+4t+2 = \underline{4t+4} \\ 2D(D+6)[x] + 2D(D+2)[y] = 2D[2t+1] = \underline{4t} \end{cases}$$

④ $(D^2-4D+4)[x] = 4t$

$$r = 2, 2$$

$$x_h = c_1 e^{2t} + c_2 t e^{2t}$$

$$x_p = At + B$$

$$x_p' = A \text{ ①}$$

$$x_p'' = 0 \text{ ①}$$

$$x_p'' - 4x_p' + 4x_p = -4A + 4At + 4B = \underline{4At + (-4A + 4B)} = 4t$$

$$4A = 4 \rightarrow A = 1$$

$$-4A + 4B = 0 \rightarrow B = A = 1$$

$$x = \underline{t + 1 + c_1 e^{2t} + c_2 t e^{2t}}$$

$$\Rightarrow \begin{cases} (D+6)(3D+2)[x] + (D+6)(2D)[y] = (D+6)[2t+1] = 2+12t+6 = \underline{12t+8} \\ (3D+2)(D+6)[x] + (3D+2)(D+2)[y] = (3D+2)[2t+1] = 6+4t+2 = \underline{4t+8} \end{cases}$$

$$(D^2-4D+4)[y] = -8t$$

$$y_h = k_1 e^{2t} + k_2 t e^{2t}$$

$$y_p = Ct + E$$

$$y_p'' - 4y_p' + 4y_p = 4Ct + (-4C + 4E) = -8t$$

$$4C = -8 \rightarrow C = -2$$

$$-4C + 4E = 0 \rightarrow E = C = -2 \text{ ①}$$

$$y = \underline{-2t - 2 + k_1 e^{2t} + k_2 t e^{2t}}$$

$$(3D+2)[x] + 2D[y] = \begin{array}{l} 3 + 6c_1 e^{2t} + 6c_2 t e^{2t} \\ + 3c_2 e^{2t} \\ + 2t + 2 + 2c_1 e^{2t} + 2c_2 t e^{2t} \\ - 4 + 4k_1 e^{2t} + 4k_2 t e^{2t} \\ + 2k_2 e^{2t} \end{array} \text{ ④}$$

$$= \underline{2t + 1 + (8c_1 + 3c_2 + 4k_1 + 2k_2)e^{2t} + (8c_2 + 4k_2)t e^{2t}} \text{ ④}$$

$$x = t + 1 + c_1 e^{2t} + c_2 t e^{2t}$$

$$y = \underline{-2t - 2 + (-2c_1 + \frac{1}{4}c_2)e^{2t} - 2c_2 t e^{2t}}$$

$$8c_1 + 3c_2 + 4k_1 + 2k_2 = 0$$

$$8c_2 + 4k_2 = 0 \rightarrow \underline{k_2 = -2c_2}$$

$$8c_1 - c_2 + 4k_1 = 0 \rightarrow \underline{k_1 = -2c_1 + \frac{1}{4}c_2}$$